Simple model

- Firms buy GDP for investment and rent labor at wage rate w (firms are price-takers)
- The price of capital goods is $P_K$ (for simplicity you can assume that $P_K = 1$)
- Firms have access to financial market; discount rate = $r$ (future production is worth less than today's production)
Desired capital stock

- To simplify, assume zero depreciation and no inflation
- Firms maximize profit

\[ \Pi = \sum_{t=0}^{\infty} \frac{1}{(1 + r)^{t+1}} \left( AF_s(K_s, L_s) - wL_s - P_s I_s \right) \]

- subject to: \( K_{s+1} = K_s + I_s \)

Solution

Step 1: formulate Lagrangian:

\[ L = \sum_{s=0}^{\infty} \frac{1}{(1 + r)^{s+1}} \left\{ AF(K_s, L_s) - P_s I_s - wL_s \right\} + \sum_{s=0}^{\infty} \lambda_s (K_s + I_s - K_{s+1}) \]
Solution

• Step 2: maximize with respect to capital and investment to get:
  • $AF'(K_s) = rP_k$

This says that in equilibrium, marginal product of capital equals the return on alternative assets.

Note: when you calculate FOC with respect to $K_s$, it appears twice in the budget constraint: in time $s$ as $K_s$ and in time $s-1$ as $K_{s+1}$

Problems with the simple model

• Strange predictions:
  • A change in $r$ (or $A$) causes and an immediate jump in desired level of capital and an infinite jump in investment
  • No room for expectations
Modification

- Introduce costs of investment
- Turning I into K takes effort and time: $\Phi(I/K)$ is the cost of investment
- The more firm invests at the same time, the more costly it is
- Firms learn from past investment, so the more capital they have, the less costly it is to install new capital
- The cost is never negative (disinvestment is also costly)
- $\Phi(0) = 0; \Phi'(0) = 0$
- Example: $\frac{\chi I^2}{2K}$

Solution

- Now the Lagrangian is:

$$L = \sum_{s=t}^{\infty} \frac{1}{(1+r)^{s-t}} \left[ AF(K_s, I_s) - P_s I_s - \phi(I/K) - wI_s \right] + \sum_{s=t}^{\infty} \lambda_s(K_s + I_s - K_{s+1})$$

- Before we solve it, let’s modify it slightly
- Recall that $\lambda$ is the shadow price of capital at time $t$ (at the starting point)
- Introduce $q = \text{the shadow price at the current time } s$, hence $q_s = (1+r)^{s-t} \lambda_t$
Solution

The Lagrangian becomes:
\[ L = \sum_{s=t}^{\infty} \frac{1}{(1+r)^{s-t}} \left[ AF(K_s, L_s) - P_s I_s - wL_s - \phi(I_s / K_s) + q_s (K_s + I_s - K_{s-1}) \right] \]

Maximize with respect to capital and investment to get the first order conditions (FOC):
\[ \begin{align*}
1 & \quad P_K' + \phi_I' = q \\
2 & \quad \frac{1}{(1+r)^{s-t}} (F'_K + \phi'_K + q_s) - \frac{q_{s-1}}{(1+r)^{s-1-t}} = 0
\end{align*} \]

Solution

- Rearrange the 2nd FOC to get
\[ (F'_K + \phi'_K + q_s) - (1+r)q_{s-1} = 0 \]
\[ (F'_K) = q_{s-1} + rq_{s-1} - \phi'_K - q_s = rq_{s-1} - \Delta q_s - \phi'_K \]
\[ (F'_K) + \Delta q_s + \phi'_K = rq_{s-1} \]
Solution

- Don’t forget: transversality condition must hold as well!
- Is says that in the „last period” – infinity – the amount of capital times the shadow price must be zero. So in the last period the firm either has no capital, or its value is zero

\[
\lim_{s \to \infty} \frac{1}{(1 + r)^{s-t}} q_s k_s = 0
\]

What does that mean?

First FOC says that when the firm is optimizing, the marginal cost of investment (i.e. buying and installing an additional unit of capital) has to be equal to the shadow price of capital.

**Note:** From first FOC we see that investment ratio \( I/K \) is a function of \( q \).

Second FOC says that when the firm is optimising, marginal gain from additional unit of capital plus any increase in the shadow price plus a decrease in the costs of investment has to be equal to the opportunity cost of additional unit of capital (the „lost“ interest rate)

**Note:** a firm has to solve two problems – how much capital stock should it have and how much to invest in a given period of time.
Solution: a phase diagram

- To study the dynamic behavior of firms, draw a diagram, with $k$ on horizontal axis and $q$ on a vertical axis.
- Find the locus of point, where the amount of capital is stable and the value of $q$ is stable.
- Capital stock is unchanging, when investments are zero (recall that we assumed that depreciation is zero), i.e. $\Phi'_i$ is zero.
- This is true when $q = P_K$ (or 1 if, $P_K = 1$).
- The gain from additional unit of capital is equal to the cost of investment.

Solution: a phase diagram

- From the 2 FOC, the value of $q$ is stable, if the marginal cost of capital is equal to its marginal product plus $\Phi'_K$.
- Note that marginal product of capital is decreasing.
- That implies a phase diagram like:
Phase diagram

\[ \Delta K > 0 \]

\[ \Delta K = 0 \]

\[ \Delta K < 0 \]

Phase diagram

\[ \Delta q > 0 \]

\[ \Delta q < 0 \]

\[ \Delta q = 0 \]
Phase diagram

- Saddle-point stable system
- The system has a unique path (SS) that converges to the steady-state.
- Specifically, for any starting capital stock $K_t$, there is only one value of $q_t$ that places the firm on the stable adjustment path.
- The unstable paths are paths that satisfy the FOC but do not satisfy the transversality condition (hence are not optimal)
- NE part: both $K$ and $q$ grow forever – violation of transversality condition
- In SW part: both $K$ and $q$ fall – even turn negative – violation of transversality condition
Steady state

- $q$ is equal to price of capital and doesn’t change.
- That implies (see 2nd FOC) that the marginal product of capital is equal to the interest rate times the price of capital (if its one, then $MPK = r$).

The dynamics of the ss path

- If the firm starts with a capital stock below $K^*$, the marginal benefit of capital is lower than the cost of capital.
- Attempting to raise capital instantly is ruled out by high installation costs. Capital rises gradually.
- The high marginal product of capital is reflected in high value of $q$.
- As capital increases, the value of $q$ falls. As $q$ falls to one, investment declines to zero.
The effects of an increase in output

- A firm is in the steady-state.
- \( Y \) rises permanently. The \( \Delta q=0 \) line shifts up. \( Q \) jumps up to the new ss path and \( K \) gradually increases to the new stead-state value (i.e. a transitory increase in investment)
- As \( K \) grows, \( q \) converges to \( P_K \)

The effects of an increase in interest rate

- A firm is in the steady-state.
- \( r \) rises permanently. The \( \Delta q=0 \) line shifts down (and changes slope). \( Q \) jumps down to the new ss path and \( K \) gradually decreases to the new stead-state value
- As \( K \) declines, \( q \) converges to \( P_K \)
• Increases in $A$, decreases in $T$
• The $\Delta q=0$ line shifts up, capital stock increases, investment increases and then converges to zero

A corporate tax cut…
Empirics

- Q is all that matters for investment decisions
- Yet, q is not really a price, which we can directly observe
- Unless we can relate q to some observable variable, the theory has little empirical content
- Hayashi showed that under plausible assumption shadow price is equal to the ratio of the (stock) market value of the firm, divided by the replacement cost of capital.

Continuous time

\[
\max_{K} \int_{t}^{t+\infty} e^{-rt} \left[ AF(K_t, L_t) - P_t I_t - L_t - \varphi \left( \frac{L_t}{K_t} \right) \right] dt
\]

subject to:

\[
K_t - I_t = 0
\]
Solution

- Formulate Hamiltonian:
  \[ H = e^{\alpha t} AF(K_t, L_t) - wL_t - P_K I_t - \Phi(I/K) + \lambda I_t \]
- FOC:
  \[ \frac{\delta H}{\delta K} = -\lambda \]
  \[ \frac{\delta H}{\delta I} = 0 \]
Summers (1981) test